Discrete Mathematics and Its Applications 2 (CS147)

Lecture 2: Big-O notation

Fanghui Liu

Department of Computer Science, University of Warwick, UK

How to analyse runtimes of algorithms?

Preliminaries: Big-O notation

Basic definitions

Consider any two functions $f(n)$ and $g(n)$ satisfying

$$f(n) > 0, g(n) > 0 \quad \text{for all positive integer } n$$

- $g(n) \in \mathcal{O}(f(n))$ [**Big O** of $f(n)$] if there exist constants $c > 0$ and N such that $g(n) \leq cf(n)$ for all $n > N$.

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- $g(n) \in \Omega(f(n))$ [**Big Omega** of $f(n)$] if there exist constants $c > 0$ and N such that $g(n) \geq cf(n)$ for all $n > N$.
- $g(n) \in \Theta(f(n))$ [**Big Theta** of $f(n)$] if $g(n) \in \mathcal{O}(f(n))$ and $g(n) \in \Omega(f(n))$.
  $\Leftrightarrow$ there exist constants $c_1, c_2 > 0$ and N such that $c_1 f(n) \leq g(n) \leq c_2 f(n)$ for all $n > N$.

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- $g(n) \in o(f(n))$ [**little o** of $f(n)$] if for every $c > 0$, there exists an N such that $g(n) \leq cf(n)$ for all $n > N$.

$$\Leftrightarrow \lim_{n\to\infty} \frac{g(n)}{f(n)} = 0\,.$$

- $g(n) \in \omega(f(n))$ [**little omega** of $f(n)$] if for every $c > 0$, there exists an N such that $g(n) \geq cf(n)$ for all $n > N$.

$$\Leftrightarrow \lim_{n\to\infty} \frac{g(n)}{f(n)} = \infty\,.$$

Intuitions

- $g(n) \in \mathcal{O}(f(n))$ means $g \leq f$ "asymptotically".
- $g(n) \in \Omega(f(n))$ means $g \geq f$ "asymptotically".
- $g(n) \in \Theta(f(n))$ means $g = f$ "asymptotically".
- $g(n) \in o(f(n))$ means $g < f$ "asymptotically".
- $g(n) \in \omega(f(n))$ means $g > f$ "asymptotically".

Examples

Example

The function n is in $\mathcal{O}(n^3)$.

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Set $c = 1$, $N = 1$, then $n \leq cn^3$ for all $n \geq N$. □

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The function n^3 is not in $\mathcal{O}(n)$.

Proof.

Suppose that there exist constants $c > 0$ and N such that $n^3 \leq cn$ for all $n > N$. But we have $n^3 > cn$ for all $n > \sqrt{c}$. This leads to a contradiction, and concludes the proof. □

Abusing the equals sign

- We write $g(n) = \mathcal{O}(f(n))$ to denote $g(n) \in \mathcal{O}(f(n))$.
- Consider the statement $\mathcal{O}(n^2) + \mathcal{O}(n^3) + 1 = \mathcal{O}(n^3)$. Formally, this means that:

Statement

for all $f(n) \in \mathcal{O}(n^2)$ and $g(n) \in \mathcal{O}(n^3)$, we have $f(n) + g(n) + 1 \in \mathcal{O}(n^3)$.

  - Example: $\frac{n^2}{2} + \frac{n^3}{4} = \mathcal{O}(n^2) + \mathcal{O}(n^3) = \mathcal{O}(n^3)$

Principles: useful simplifications

- If $g(n) = \mathcal{O}(f(n))$, we have

$$\mathcal{O}(f(n)) + \mathcal{O}(g(n)) = \mathcal{O}(f(n)) \,.$$

  - Example: $2n^4 + n^7/3 = \mathcal{O}(n^7)$.
- If c is a constant, we have

$$\mathcal{O}(cf(n)) = \mathcal{O}(f(n)) \,.$$

  - Example 1: $\mathcal{O}(5n^2) = \mathcal{O}(n^2)$.
  - Example 2: $\mathcal{O}(\log_a n) = \mathcal{O}(\log_b n)$ if $a, b > 0$ are constants.

Example

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Given $f(n) = 2^{\sqrt{\ln n}}$ and $g(n) = n^{0.0001}$, check $f(n) = \mathcal{O}(g(n))$ and $f(n) \neq \Omega(g(n))$.

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Proof.

Take the logarithmic operation of $f(n)$ and $g(n)$...
Updated: We can prove it by the definition. We prove $f(n) = \mathcal{O}(g(n))$ as an example, that means, we need to find $c > 0$ and $n > N$ such that $f(n) \leq cg(n)$

$$2^{\sqrt{\ln n}} \leq cn^{0.0001} \quad \Leftrightarrow \sqrt{\ln n}\ln 2 \leq \ln c + 0.0001 \ln n \quad [\text{taking } c \geq 1 \text{ for nonnegativity}]$$

$$\Leftrightarrow \sqrt{\ln n}\ln 2 \leq 0.0001 \ln n \quad [\text{taking } c = 1 \text{ for simplicity}]$$

$$\Leftrightarrow \sqrt{\ln n} \geq 10^4 \ln 2 \quad \Leftrightarrow n \geq e^{10^8 (\ln 2)^2} := N\,.$$

We conclude the proof by taking $c = 1$ and $N := e^{10^8 (\ln 2)^2}$. □

Comparison

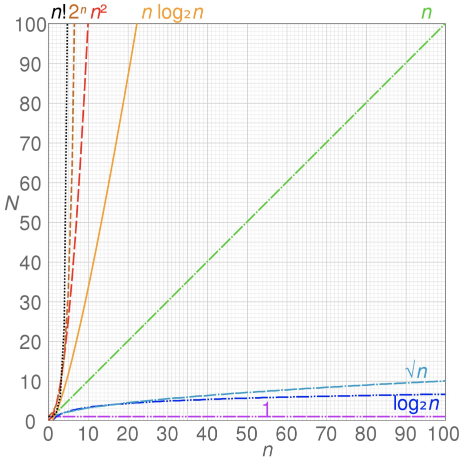

Figure: source from `https://en.wikipedia.org/wiki/Big_O_notation`.

Principles: using limits

- If $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ exists (and is finite), then $f(n) = \mathcal{O}(g(n))$.

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- If $\lim_{n\to\infty} \frac{f(n)}{g(n)}$ exists (and is finite), then $f(n) = \mathcal{O}(g(n))$.
- We can often apply L'Höpital's rule to calculate this limit

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)} .$$

Remark: Remember the condition when using the L'Höpital's rule.

Example (Updated)

Consider two functions $f(n) = 2^n$ and $g(n) = 3^n$, we have $f(n) = \mathcal{O}(g(n))$ and $f(n) \neq \Omega(g(n))$.

Asymptotic notations and summations (I)

- Consider an geometric series $f(n) = \sum_{i=1}^{n} x^i = \frac{1-x^{n+1}}{1-x}$.
  - If $x > 1$, then $f(n) = \Theta(x^n)$.
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- Dealing with an arithmetic series with two constants a, b

$$\sum_{i=1}^{n}(ai+b) = \sum_{i=1}^{n} ai + \sum_{i=1}^{n} b = a\frac{n(n+1)}{2} + bn = \Theta(n^2) + \Theta(n) = \Theta(n^2)\,.$$

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- Dealing with an harmonic series:

$$\sum_{i=1}^{n}\frac{2n}{i} = \Theta\left(\sum_{i=1}^{n}\frac{n}{i}\right) = \Theta\left(n\sum_{i=1}^{n}\frac{1}{i}\right) = \Theta(n\log n)\,.$$

[recall harmonic series:] $\sum_{i=1}^{k}\frac{1}{i} = \ln k + \gamma + \epsilon_k$ with the Euler–Mascheroni constant $\gamma \approx 0.577$ and $\epsilon_k \approx \frac{1}{2k}$.

Asymptotic notations and summations (II)

- Bounding parts of the sum: consider $f(n) = \sum_{i=1}^{n} i^3$
  - upper bound

$$f(n) = \sum_{i=1}^{n} i^3 \le \sum_{i=1}^{n} n^3 = \mathcal{O}(n^4)\,.$$

  - lower bound

$$f(n) = \sum_{i=1}^{n} i^3 \ge \sum_{i=\frac{n}{2}}^{n} n^3 \ge \sum_{i=\frac{n}{2}}^{n} (\frac{n}{2})^3 = \Omega(n^4)\,.$$

Accordingly, we have $f(n) = \Theta(n^4)$.

Asymptotic notations and summations (III)

- Consider the function $f(n) = \sum_{i=1}^{n} a^i i$
  - if $a > 1$, then $f(n) = \Theta(a^n n)$.
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  - if $a = 1$, then $f(n) = \Theta(n)$.
  - if $0 < a < 1$, then $f(n) = \Theta(1)$.
- Integration: If $f(n)$ is non-decreasing in n, then

$$\int_{a-1}^{b} f(x)\mathrm{d}x \leq \sum_{i=a}^{b} f(i) \leq \int_{a}^{b+1} f(x)\mathrm{d}x \,,$$

where $a < b$ are integers.

References I